Interface dynamics in a uniaxial anisotropic *n*-vector model

Hiroki Tutu
Department of Physics, Kyushu University 33, Fukuoka 812-81, Japan
(Received 24 September 1997)

We investigate interface dynamics of a nonconserved n-vector model with a positive uniaxial anisotropy and derive its evolution equation. The positive uniaxial anisotropy makes the symmetry of the order parameter lower, and the medium forms an interface structure. In a weak anisotropic case the interface has an additive order parameter having $(n-1)$ degrees of freedom with $O(n-1)$ symmetry. The evolution equation is incorporated with the fluctuation associated with additive $(n-1)$ degrees of freedom on the interface, and the resultant form has the extended form of the one derived by Allen and Cahn for the scalar order parameter system. [S1063-651X(98)08003-9]

PACS number(s): 05.70.Ln, 64.60.Cn, 75.60.Ch, 75.30.Gw

I. INTRODUCTION

The dynamics in which the interface plays an important role has had much attention in the context of magnetic systems [1], binary mixtures [2], reaction-diffusion systems, and liquid crystal systems [3]. Some familiar examples are what appear in ordering process in a ferromagnet or binary mixtures. In such systems the order parameter is a scalar field variable and the free energy has two local minima, which are produced by temperature change.

The interface formed in a scalar order parameter system has no particular structure except curvature describing its local geometry when the embedded space is isotropic and homogeneous. Furthermore the dynamics minimizes its surface energy which is driven by curvature in the ordering process in purely dissipative cases. In an extension of these systems we consider a situation where the interface has an additive vector order parameter with $O(n-1)$ symmetry, in addition to curvature. One such system is realized in the n-vector model with the positive uniaxial anisotropy axis in the order parameter space.

The n-vector model describes a medium whose order parameter has $O(n)$ symmetry. In this model the rotation in the physical space and the rotation in the inner space of order parameter are independent [4]. The model describes the XY-spin system for $n=2$ and a Heisenberg spin system for $n=3$, for example. The nonequilibrium dynamical study on this model was developed by several authors in the frame of the ordering process [5].

The introduction of uniaxial anisotropy to the n-vector model reduces the symmetry of the order parameter to $O(n-1)$, $O(n-1)\times Z_2$, and Z_2 symmetry, which, respectively, correspond to the strong negative anisotropy, (positive or negative) weak anisotropy, and strong positive anisotropy. The sign is defined in the next section. Due to the symmetric property, if positive anisotropy is induced, the system possesses interfaces which separates two stable states. Especially if the anisotropy is weak, the interface has additive $(n-1)$ degrees of freedom whose symmetry is $O(n-1)$. The interest is in how to describe the local dynamics which is responsible for evolutions of interfaces.

In previous works we studied both one- and two-dimensional anisotropic XY-spin systems $(n=2)$ in nonconserved systems [6,7]. In both works we derived the evolution equation for the domain wall, the so-called Bloch wall, having chirality. In one dimension, we showed that the chirality brings about either a repulsive or attractive interaction between neighboring walls in contrast to the Ising walls which always bring an attractive interaction. In two dimensions, the chirality does not affect the global features of the dynamics, such as the growth law for the average domain size, and only slightly affects the interface shape.

The fundamental purpose of the present paper is to show the derivation of the evolution equation of the interface in the n-vector model with positive uniaxial anisotropy. This is a straight extension of our previous work for the two-dimensional anisotropic XY-spin system. Results can be generalized to any space dimensions and number of components.

This paper is organized as follows. In Sec. II the model is presented. In Sec. III we obtain the fundamental equations controlling the interface dynamics for this system, and then discuss their features in Sec. IV. Finally we summarize our results in Sec. V.

II. THE MODEL

Let us consider the Ginzburg-Landau (GL) type free energy for the n-vector system with anisotropy in d-dimensional space,

$$H\{\boldsymbol{\psi}\}=\int d^d\mathbf{r}\left\{V(|\boldsymbol{\psi}|)-\frac{1}{2}\sum_{\alpha,\beta}\psi^\alpha\gamma_{\alpha\beta}\psi^\beta+\frac{1}{2}\sum_{j,\alpha}\partial_j\psi^\alpha\partial_j\psi^\alpha\right\}, \tag{2.1}$$

where $\boldsymbol{\psi}$ is the n-component vector order parameter $\boldsymbol{\psi}=(\psi^1,\psi^2,\dots,\psi^n)$, $\gamma_{\alpha\beta}$ is a symmetric tensor which represents the anisotropy, and $V(x)=(1-x^2)^2/4$. Since we consider uniaxial anisotropy, $\gamma_{\alpha\beta}$ is chosen in such a way that the first component of the order parameter is the principal axis without loss of generality, i.e., $\gamma_{11}=\gamma$, $\gamma_{\alpha\alpha}=-\gamma$ $(\alpha\neq 1)$, $\gamma_{\alpha\beta}=0$ $(\alpha\neq\beta)$. This is the simplest model for the n-vector model with uniaxial anisotropy considering only isotropic elastic energy.

For the time evolution we consider the nonconserved purely dissipative dynamics

$$\partial_t \psi^\alpha = -\frac{\delta H}{\delta \psi^\alpha} = (1-|\boldsymbol{\psi}|^2)\psi^\alpha + \gamma_{\alpha\alpha}\psi^\alpha + \nabla^2 \psi^\alpha. \quad (2.2)$$

Here we ignore thermal noise and concern ourselves with the system which is quenched from above the critical temperature.

Let us first investigate the uniform solution of Eq. (2.2). For positive γ $(0<\gamma<1)$, there exist stable uniform solutions $\boldsymbol{\psi} = \pm\sqrt{1+\gamma}\mathbf{u}_1$, where $\mathbf{u}_1 = (1,0,\ldots,0)$, and unstable uniform solutions $\boldsymbol{\psi}=0$, $\boldsymbol{\psi} = \pm\sqrt{1-\gamma}\mathbf{u}_\perp$, where $\mathbf{u}_\perp$ is a unit vector perpendicular to $\mathbf{u}_1$. For negative γ $(-1<\gamma<0)$, the $\mathbf{u}_1$ direction becomes unstable, and $\mathbf{u}_\perp$ directions, which represent any direction perpendicular to the $\mathbf{u}_1$ direction, are stable. For the region $|\gamma|>1$ one can treat the system as a one-component system for $\gamma>1$ and a $(n-1)$-component system for $\gamma<-1$.

Next we consider the nonuniform equilibrium solution. In a two-component system, two types of domain wall solutions, the Bloch and Ising (Néel) walls, are known. For the Ising wall the amplitude of the order parameter vanishes at the wall, while for the Bloch wall, it has a nonvanishing structure, the so-called chirality [1].

For a positive γ we find the same types of solutions as the Bloch wall and the Ising wall even in the n-component system. In the one-dimensional space they are respectively represented as

$$\boldsymbol{\psi}_{I+} = \sqrt{1+\gamma}\tanh[\sqrt{(1+\gamma)/2}x]\mathbf{u}_1, \quad (2.3)$$

$$\boldsymbol{\psi}_{B+} = X_B \tanh(x/\xi_B)\mathbf{u}_1 + Y_B \operatorname{sech}(x/\xi_B)\mathbf{u}_\perp, \quad (2.4)$$

where $X_B=\sqrt{1+\gamma}$, $Y_B=\sqrt{1-3\gamma}$, and $\xi_B = 1/\sqrt{2\gamma}$. Here the direction of $\mathbf{u}_1$ and $\mathbf{u}_\perp$ in the order parameter space has no reference to the space coordinate system (x axis).

In order to see the relative stability of two wall solutions we show the free energy per one wall in one-dimensionalized space. Substituting solutions (2.3) and (2.4) into Eq. (2.1) and carrying out the integration, we obtain corresponding free energies for Ising and Bloch walls as follows:

$$f_I(\gamma) = \text{const} + \frac{2\sqrt{2}}{3}(1+\gamma)^{3/2}, \quad (2.5)$$

$$f_B(\gamma) = \text{const} + \left[\frac{32}{3}\gamma^2 + \gamma(1-3\gamma)\right] \Big/ \sqrt{2\gamma}. \quad (2.6)$$

Here const has the same value for both $f_I(\gamma)$ and $f_B(\gamma)$ and it does not effect the stability of the solutions. Figure 1 represents the behavior of the two free energies. The stable wall is obtained from the lowest free energy. Therefore the Ising wall and the Bloch wall are realized in the region $\gamma>1/3$ and $0<\gamma<1/3$, respectively, and give rise to transition continuously to each other at $\gamma=1/3$ as γ is changed.

For negative γ the type of equilibrium solution is affected by the space dimension d and the number of components n. In the two-component system the solutions corresponding to the Ising and the Bloch wall again exist for all dimensionalities. Namely, solutions are obtained by the replacement $\gamma \to -\gamma(>0)$, $\mathbf{u}_1 \to \mathbf{u}_2$, $\mathbf{u}_\perp \to \mathbf{u}_1$, $(\mathbf{u}_1\cdot\mathbf{u}_2=0)$ in Eqs. (2.3) and (2.4). However, for the system with $n\geq 3$ one must take into account the spatial dimensionality. Namely, in the case $\gamma<0$ the order parameter space has isotropic $(n-1)$ components and another additive one in d-dimensional space. Thus when $n-1\leq d$ $(\gamma<0)$ there exists a solution having a defect structure. (In the O(n) model there exists defect solution when $n\leq d$ [8].) However, from the analogy of Ising and Bloch wall solutions one expects two types of solutions, one which has vanishing order parameter at the defect core or another which has nonvanishing amplitude at the core, respectively, for strong and weak anisotropic cases.

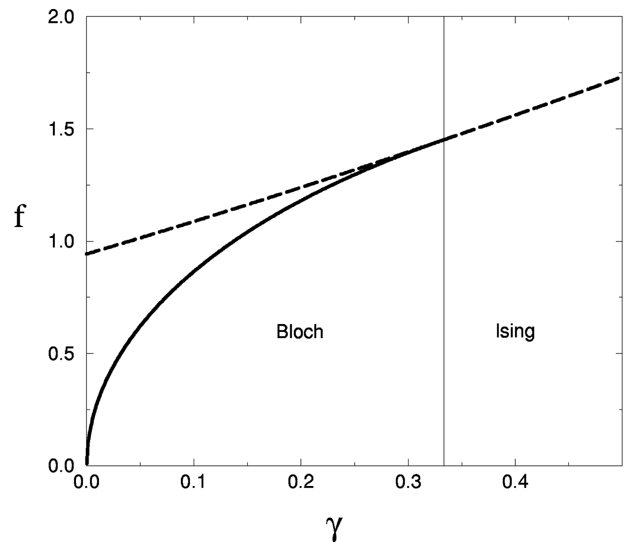

FIG. 1. The relation between anisotropy γ and free energy f per one domain wall in the one-dimensional system. The solid line denotes the free energy for the Bloch wall and the dashed line denotes the one for the Ising wall. In the region $0<\gamma<1/3$ $(1/3<\gamma)$ the Bloch (Ising) wall is more stable than the other.

For an example, one can imagine the two types of solutions in a two-dimensional three-component system as follows:

$$\boldsymbol{\psi}_{I-} = \sqrt{1+|\gamma|}\rho_I[\sqrt{(1+|\gamma|)/2}r]\mathbf{U}_\perp, \quad (2.7)$$

$$\boldsymbol{\psi}_{B-} = X_B \rho_B(r/\xi_B)\mathbf{U}_\perp + qY_B\overline{\rho_B}(r/\xi_B)\mathbf{u}_1. \quad (2.8)$$

Here $\mathbf{U}_\perp$ is a vector having rotational symmetry without the dependence on the radius r, and $\rho_{I,B}$ and $\overline{\rho_B}$ are certain functions satisfying $\rho_I(0)=\rho_B(0)=0$, $\overline{\rho_B}(0)\neq 0$, $\rho_{I,B}(\infty)=1$, and $\overline{\rho_B}(\infty)=0$, and q takes either $+1$ and -1. We have not yet obtained the explicit expression of these functions. Hence we cannot say anything whether the transition occurs at $|\gamma|=1/3$ or not.

In the next section we derive the evolution equation for the interface focusing on the regime $0<\gamma<1/3$.

III. THE INTERFACE DYNAMICS

Interface dynamics for conserved and nonconserved one-component order parameter system have been developed by Lifshitz [9], Allen and Cahn [10], and Kawasaki and Ohta [2]. In our previous study we investigated the dynamics of

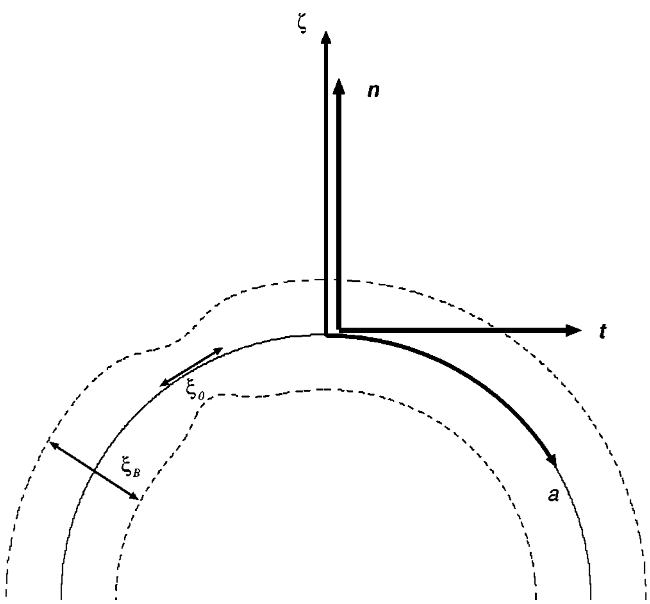

FIG. 2. A schematic figure of coordinate system parametrized on the interface. The solid and dashed lines respectively denote the interface defined by $\psi^1=0$ and its width. The coordinate system consists of the ζ axis along the n direction which is perpendicular to the interface and a which lies on the interface. Furthermore, the figure schematically represents the meanings of the two characteristic length constants denoted by ξ_B and ξ_0. ξ_B and ξ_0 are respectively the length scales perpendicular to the interface for ψ^1 and spatial variation within the interface for $\boldsymbol{\phi}$.

the interface for the anisotropic XY-spin system. As mentioned in the previous section an interface is formed in all spatial dimensions when the positive uniaxial anisotropy is induced in the n-vector model. In this paper we show that the dynamics of the interface for the anisotropic n-vector model can be obtained from a straightforward extension of our previous work.

Let us define the interface as the surface satisfying $\psi^1=0$ and introduce the curvilinear coordinate system (ζ,a). ζ represents a coordinate generated by the tangential direction to the vector field $\mathbf{n}$, which is the unit normal vector to the surface, and $a\equiv(a_1,a_2,\ldots,a_{d-1})$ represents the coordinate system within the surface. This is schematically illustrated in Fig. 2. By introducing the metric $g_\zeta=\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\zeta$, the ζ axis and the unit normal vector $\mathbf{n}$ are related via

$$\zeta=\int_{\mathbf{r}(a)}^{\mathbf{r}}\sqrt{g_\zeta}\mathbf{n}\cdot d\mathbf{r}, \tag{3.1}$$

where $\mathbf{r}(a)$ represents a point on the surface. By introducing g_ζ, one can take into account fluctuations produced by $(n-1)$ degrees of freedom for the order parameter on the surface.

The order parameter profile $\boldsymbol{\psi}$ can be well parametrized by introducing the second order parameter $\boldsymbol{\phi}\equiv\boldsymbol{\phi}(\mathbf{r},t)\equiv(0,\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_{n-1})$ with $(n-1)$ components, whose symmetry is $\mathrm{O}(n-1)$, as

$$\boldsymbol{\psi}(\mathbf{r},t)=X_B\tanh(\zeta/\xi_B)\mathbf{u}_1+Y_B\mathrm{sech}(\zeta/\xi_B)\boldsymbol{\phi}(\mathbf{r},t). \tag{3.2}$$

Substituting Eq. (3.2) into Eq. (2.2), after a straightforward calculation we obtain

$$\partial_t\zeta=\xi_B(X_B^2-Y_B^2|\boldsymbol{\phi}|^2-2g_\zeta/\xi_B^2)\tanh(\zeta/\xi_B)+\boldsymbol{\nabla}^2\zeta, \tag{3.3}$$

$$\partial_t\boldsymbol{\phi}=(1-\gamma-Y_B^2|\boldsymbol{\phi}|^2-g_\zeta/\xi_B^2)\boldsymbol{\phi}+\boldsymbol{\nabla}^2\boldsymbol{\phi} -2\tanh(\zeta/\xi_B)/\xi_B\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{\phi}. \tag{3.4}$$

These equations are not closed since we have not yet shown how g_ζ depends on such quantities as $\boldsymbol{\phi}$ or the curvature. Thus we must determine an equation for g_ζ. Differentiating both sides of Eq. (3.3) and operating the inner product with $\mathbf{n}$ to both sides, we obtain

$$\begin{aligned}\partial_t\sqrt{g_\zeta}=&(X_B^2-Y_B^2|\boldsymbol{\phi}|^2-2g_\zeta/\xi_B^2)\sqrt{g_\zeta}\mathrm{sech}^2(\zeta/\xi_B)\\&+(\mathbf{n}\cdot\boldsymbol{\nabla}^2\mathbf{n})\sqrt{g_\zeta}+\boldsymbol{\nabla}^2\sqrt{g_\zeta}+\xi_B\tanh(\zeta/\xi_B)\\&\times\mathbf{n}\cdot\boldsymbol{\nabla}(X_B^2-Y_B^2|\boldsymbol{\phi}|^2-2g_\zeta/\xi_B^2).\end{aligned} \tag{3.5}$$

In the following we use an approximation that quantities g_ζ and $\boldsymbol{\phi}$, which move together with interface, change quasistatically over space and time in the reference system to the interface. Since g_ζ is regarded as a slowly varying quantity in comparison with ζ near the interface, we can average Eq. (3.5) to extract the slowly varying quantity. Integrating Eq. (3.5) with ζ after multiplying by $\mathrm{sech}^2(\zeta/\xi_B)$ with the assumption that variables $\boldsymbol{\phi}$ and $\mathbf{n}$ are slowly varying on the interface, the above equation simplifies to

$$\partial_t\sqrt{g_\zeta}\approx C^{-1}(X_B^2-Y_B^2|\boldsymbol{\phi}|^2-2g_\zeta/\xi_B^2)\sqrt{g_\zeta}+(\mathbf{n}\cdot\boldsymbol{\nabla}^2\mathbf{n})\sqrt{g_\zeta} +\boldsymbol{\nabla}^2\sqrt{g_\zeta}, \tag{3.6}$$

where C is a numerical factor, $C^{-1}=\int\mathrm{sech}^4(y)dy/\int\mathrm{sech}^2(y)dy=2/3$.

The dynamics of the interface is described by a set of equations Eqs. (3.3), (3.4), and (3.6). Especially Eq. (3.6) describes the time development of the width of wall. Neglecting time and higher space derivative of g_ζ by applying the quasistatic approximation, g_ζ is adiabatically determined as

$$2g_\zeta/\xi_B^2\approx X_B^2-Y_B^2|\boldsymbol{\phi}|^2+C(\mathbf{n}\cdot\boldsymbol{\nabla}^2\mathbf{n}), \tag{3.7}$$

by ignoring terms higher than the second derivative of $\mathbf{n}$.

And then successively using the quasistatic assumption for Eq. (3.3), i.e., the order parameter profile is slowly varying in the reference frame, as $\partial_t\zeta+\mathbf{v}\cdot\boldsymbol{\nabla}\zeta=0$ with $\mathbf{v}\equiv\mathbf{v}(\mathbf{r},t)$ being the velocity of the frame, we obtain

$$\mathbf{v}_n\approx-\frac{1}{\sqrt{g_\zeta}}\boldsymbol{\nabla}\cdot\sqrt{g_\zeta}\mathbf{n}. \tag{3.8}$$

Furthermore, by using Eq. (3.7), Eq. (3.4) is reduced to

$$\partial_t\boldsymbol{\phi}=\frac{Y_B^2}{2}(1-|\boldsymbol{\phi}|^2)\boldsymbol{\phi}-\frac{C}{2}(\mathbf{n}\cdot\boldsymbol{\nabla}^2\mathbf{n})\boldsymbol{\phi}+\nabla^2\boldsymbol{\phi} -2\tanh(\zeta/\xi_B)/\xi_B\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{\phi}. \tag{3.9}$$

This equation becomes more convenient as we express it by the (ζ,a)-coordinate system

$$\nabla^2 \equiv (\nabla \cdot \sqrt{g_\zeta}\mathbf{n})\partial_\zeta + g_\zeta \partial_\zeta^2 + \nabla_a^2. \tag{3.10}$$

Rewriting Eq. (3.4) by using the (ζ,a)-coordinate system, we obtain

$$\frac{d}{dt}\boldsymbol{\phi} = \frac{Y_B^2}{2}(1-|\boldsymbol{\phi}|^2)\boldsymbol{\phi} - \frac{C}{2}(\mathbf{n}\cdot\nabla_a^2\mathbf{n})\boldsymbol{\phi} + \nabla_a^2\boldsymbol{\phi} + g_\zeta\partial_\zeta^2\boldsymbol{\phi} - 2\tanh(\zeta/\xi_B)/\xi_B\partial_\zeta\boldsymbol{\phi}, \tag{3.11}$$

where we introduced the time derivative $d\boldsymbol{\phi}/dt$ on the reference frame, $(d/dt)\,\boldsymbol{\phi} = \partial_t\boldsymbol{\phi} - (\nabla\cdot\sqrt{g_\zeta}\mathbf{n})\partial_\zeta\boldsymbol{\phi}$. In the nonconserved system it is reasonable to ignore the interactions among globally separated interfaces, because they decay exponentially in the distance along the ζ direction. We assume that the ζ dependence to the $\boldsymbol{\phi}$ is very small and it results in the curvature of the interface. Thus the final reduction of Eq. (3.11) is achieved by ignoring the ζ dependence of $\boldsymbol{\phi}$ in the last terms, i.e., $(\partial_\zeta\boldsymbol{\phi}\approx 0,\ \partial_\zeta^2\boldsymbol{\phi}\approx 0)$. This reduction incorporates the curvature effect by the order of the second derivative of $\mathbf{n}$, and corrections to it are obtained perturbatively. This approximation is verified when we discuss a sufficiently smooth interface.

By summarizing above results, the set of evolution equations for the interface is written as

$$\mathbf{v}_n = -\nabla\cdot\mathbf{n} - \mathbf{n}\cdot\nabla\ln(\sqrt{g_\zeta}), \tag{3.12}$$

$$= \mathrm{Tr}K - \mathbf{n}\cdot\nabla\ln(\sqrt{g_\zeta}), \tag{3.13}$$

$$\frac{d}{dt}\boldsymbol{\phi} = \frac{Y_B^2}{2}(1-|\boldsymbol{\phi}|^2)\boldsymbol{\phi} - \frac{C}{2}(\mathbf{n}\cdot\nabla_a^2\mathbf{n})\boldsymbol{\phi} + \nabla_a^2\boldsymbol{\phi}, \tag{3.14}$$

$$= \frac{2}{\xi_0^2}(1-|\boldsymbol{\phi}|^2)\boldsymbol{\phi} + \frac{C}{2}\mathrm{Tr}K^2\boldsymbol{\phi} + \nabla_a^2\boldsymbol{\phi}, \tag{3.15}$$

$$g_\zeta = 1 + \frac{\xi_B^2 Y_B^2}{2}(1-|\boldsymbol{\phi}|^2) + \frac{C\xi_B^2}{2}\mathbf{n}\cdot\nabla_a^2\mathbf{n}, \tag{3.16}$$

$$= 1 + \frac{2\xi_B^2}{\xi_0^2}(1-|\boldsymbol{\phi}|^2) - \frac{C\xi_B^2}{2}\mathrm{Tr}K^2. \tag{3.17}$$

Here we introduced the length constant $\xi_0 = 2/Y_B = 2(1-3\gamma)^{-1/2}$, which represents a length scale within the interface and estimates the scale of defect core (see Fig. 2). For the alternative expression in the second line of each equation we have used the relations $-\nabla\cdot\mathbf{n} = \mathrm{Tr}K$ [11] and $\mathbf{n}\cdot\nabla^2\mathbf{n} = -\nabla\mathbf{n}\cdot\nabla\mathbf{n} = -\mathrm{Tr}K^2$ with the curvature tensor $K_{\alpha\beta} = K_{\beta\alpha} = -\mathbf{t}_\alpha\cdot(\partial\mathbf{n}/\partial a_\beta) = -\mathbf{t}_\beta\cdot(\partial\mathbf{n}/\partial a_\alpha)$, where $\mathbf{t}_\alpha$ is the unit vector directed along the coordinate a_α [12]. Here the $\mathrm{Tr}K$, which we simply call curvature below, is equivalent to the mean curvature multiplied by $(d-1)$.

The above equations describe the reduced dynamics of the interface coupled with the dynamics of the additional order parameter $\boldsymbol{\phi}$. The results are reasonable because they reveal the Allen-Cahn result for the evolution of the interface in the absence of the fluctuation of $\boldsymbol{\phi}$, or one obtains the usual time-dependent Ginzburg-Landau equation for $\boldsymbol{\phi}$ in the flat interface $K=0$, so both minimize the free energy.

IV. DISCUSSION

Let us discuss the implications of the results in detail. We are mainly interested in the effect of dynamical coupling between the interface and the fluctuation of the order parameter $\boldsymbol{\phi}$. However, the motion of the interface and the evolution of $\boldsymbol{\phi}$ are almost separable, indeed the coupling is negligible for the interface having an almost flat curve and an ordered $\boldsymbol{\phi}$ from Eqs. (3.15) and (3.17). The coupling may be considerable at critical regions. Hence we consider two limited situations (i) $\xi_0 \gg \xi_B$, (ii) $\xi_0 \ll \xi_B$, by noting that the system has two length constants, the width of the interface ξ_B and the width of wall ξ_0 produced by the order parameter $\boldsymbol{\phi}$ (Fig. 2).

Before proceeding to the above subject we give a summary of more useful results. First Eq. (3.15) leads to a relation $(2/\xi_0^2)(1-|\boldsymbol{\phi}|_0^2) + (C/2)\mathrm{Tr}K^2 \sim 0$ for the regions $|\boldsymbol{\phi}| \approx |\boldsymbol{\phi}_0|\ (\neq 0)$, ignoring the explicit time dependence of $\boldsymbol{\phi}$. Thus we obtain

$$|\boldsymbol{\phi}|^2 \sim \begin{cases} 1 + \dfrac{C\xi_0^2}{4}\mathrm{Tr}K^2 & \text{for } |\boldsymbol{\phi}| \sim |\boldsymbol{\phi}_0|, \\ 0 & \text{for defects.} \end{cases} \tag{4.1}$$

Then, by applying Eq. (4.1) to Eq. (3.17), g_ζ is calculated as

$$g_\zeta \sim \begin{cases} 1 - C\xi_B^2\mathrm{Tr}K^2 & \text{for } |\boldsymbol{\phi}| \sim |\boldsymbol{\phi}_0|, \\ 1 + \dfrac{2\xi_B^2}{\xi_0^2}\left(1 - \dfrac{C\xi_0^2}{4}\mathrm{Tr}K^2\right) & \text{for } |\boldsymbol{\phi}| \sim 0 \end{cases} \tag{4.2}$$

for each place on the interface.

Case (i). Consider the situation slightly below $\gamma = 1/3$, that is, the transition point of the wall structure. In this case since g_ζ has less dependence on $\boldsymbol{\phi}$ from Eq. (4.2), the velocity of the interface also almost independents on $\boldsymbol{\phi}$, and results in $\mathbf{v}_n \sim \mathrm{Tr}K$, which is the familiar form for a nonconserved one-component system, by neglecting the higher order curvature corrections. On the other hand, Eq. (4.1) shows the divergence of the amplitude of $\boldsymbol{\phi}$ as ξ_0 becomes infinity. However, such a divergence has no significant meaning. Instead, utilizing the original order parameter (3.2) rather than $\boldsymbol{\phi}$, we obtain the order parameter profile

$$\boldsymbol{\psi} \sim X[\zeta/\xi_B]\mathbf{u}_1 + \sqrt{Y_0^2 + C\mathrm{Tr}K^2}\,\mathrm{sech}(\zeta/\xi_B)\mathbf{u}_\perp. \tag{4.3}$$

This result implies that the interface fluctuation leads to the suppression of the transition from the Bloch wall to the Ising wall, since even under the transition point there remains the amplitude of the Bloch wall, which is represented by $\sqrt{C\mathrm{Tr}K^2}\mathbf{u}_\perp$ in comparison with the one-dimensional solution (2.4).

Case (ii). When γ is slightly below zero, the case is divided into two cases according to the interrelation between the space dimensionality and the number of components. For $n > d\ (d \neq 1)$, $|\boldsymbol{\phi}|^2 \sim 1$ within almost all of the interfaces since there are no stable defects. From Eq. (4.2) hence one again finds the relation $\mathbf{v}_n \sim \mathrm{Tr}K$ by assuming the condition $|\xi_B \mathrm{Tr}K| \ll 1$ for sufficiently smooth interfaces. On the contrary, in the case where $1 < n \leqslant d$, there exist stable defects, whose core size is approximately ξ_0. One may expect a slight correction to the velocity of the interface since the defect alters the width of the wall, which is interpreted as a *masslike* quantity. The second line of Eq. (4.2) is written in $g_\zeta \sim 1 + (2\xi_B^2/\xi_0^2)[1-(C\xi_0^2/4)\mathrm{Tr}K^2] \sim (2\xi_B^2/\xi_0^2)[1-(C\xi_0^2/4)\mathrm{Tr}\,K^2]$ for $\boldsymbol{\phi} \sim 0$ (defects). Substituting both form of g_ζ into Eq. (3.13), with the assumption $|\xi_B \mathrm{Tr}K| \ll 1$, we obtain

$$\mathbf{v}_n \sim \begin{cases} \mathrm{Tr}K + C\xi_B^2 \mathrm{Tr}K^3 & \text{for } \boldsymbol{\phi} \sim \boldsymbol{\phi}_0, \\ \mathrm{Tr}K + \dfrac{C}{4}\xi_0^2 \mathrm{Tr}K^3 & \text{for } \boldsymbol{\phi} \sim 0 \end{cases} \tag{4.4}$$

for both part of the interface, where we have used the relation $\mathbf{n}\cdot\boldsymbol{\nabla}\mathrm{Tr}K^2 = 2\mathrm{Tr}K^3$ (see the Appendix). For the regime $|\xi_B \mathrm{Tr}K| \sim 1$ the above expansion is not valid, and the interface picture loses its validity and instead one must use the defect picture.

Equation (4.4) shows the $|\boldsymbol{\phi}|$ dependence of the velocity of the interface. However, its effect is second order and does not cause any instability of the interface. From these considerations we again find the familiar result for the average growth law of the characteristic length scale as $l \sim t^{1/2}$.

V. SUMMARY

We obtained the interface equation for the n-vector system having positive uniaxial anisotropy in the purely dissipative and nonconserved situations by ignoring the external thermal fluctuation. Our attention is mainly focused on the behavior of the interface velocity and its dependence on the inner degrees of freedom. In the isotropic space the main contribution to the interface velocity is the force due to the surface energy, which controls the global dynamics even for this system.

In the preceding section we discussed the interface dynamics in the nearly critical case in the mean field level. We showed that the width of the interface depends on the state of the additive order parameter, and that the interface velocity also depends on it, with the second order correction of the curvature, through the dependence for the width of the interface. Although this predicts the shape dynamics of the interface connected to the additive order parameter, it is difficult to observe such dynamics over long times, especially in the purely dissipative dynamics without external noise. This is because in such dynamics the times in which the system is in criticality decreases as the system proceeds to an ordered state.

In the future our effort will be devoted to account for effects which were omitted in the present study, such as the external noise, the anisotropy of the elasticity, and others.

ACKNOWLEDGMENTS

The author is grateful to H. Fujisaka for valuable discussions and a critical reading of the manuscript. He also thanks H. Furukawa for a critical reading and helpful suggestions.

APPENDIX

Since the curvature tensor K_{ij} is defined in the second fundamental form in terms of differential geometry [12], the formula $\mathbf{n}\cdot\boldsymbol{\nabla}\mathrm{Tr}K^2 = 2\mathrm{Tr}K^3$ can be derived by considering a couple of second fundamental forms for two surfaces slightly different along the normal direction to one surface.

We define a point on a surface as $\mathbf{r}(a)$ and another point $\mathbf{r}'(a) = \mathbf{r}(a) + h\mathbf{n}$, which represents a point on the surface differed by h along the normal direction $\mathbf{n}$ to the first surface. Here we use the orthogonal coordinate system $a \equiv (a_1, a_2, \ldots)$ within the surface for simplicity.

The second fundamental form is defined by

$$\begin{aligned} \mathrm{II} \equiv -d\mathbf{n}\cdot d\mathbf{r}, \quad &= -\sum_{ij} \frac{\partial \mathbf{n}}{\partial a_i}\cdot\frac{\partial \mathbf{r}}{\partial a_j} da_i da_j, \\ &= \sum_{ij} \mathbf{n}\cdot\frac{\partial^2 \mathbf{r}}{\partial a_i \partial a_j} da_i da_j, \quad \equiv \sum_{ij} K_{ij} da_i da_j, \end{aligned} \tag{A1}$$

using the differential form. Applying this formula to the surface $\mathbf{r}'(a)$, one obtains

$$\begin{aligned} \mathrm{II}' &= -d\mathbf{n}\cdot d\mathbf{r} - h d\mathbf{n}\cdot d\mathbf{n}, \\ &= \sum_{ij} \left[K_{ij} - h\sum_k K_{ik}K_{kj} \right] da_i da_j, \end{aligned} \tag{A2}$$

for that point. Thus we obtain the curvature tensor $K'_{ij} = K_{ij} - h\Sigma_k K_{ik}K_{kj}$ at the point $\mathbf{r}'(a)$. Therefore we obtain the relation $|\mathbf{n}\cdot\boldsymbol{\nabla}\mathrm{Tr}K| = |\mathrm{Tr}K^2|$ except the sign in the limit $h \to 0$ from the definition of the derivative. The standard definition of the sign of curvature is taken as it increases along the direction of the normal vector $\mathbf{n}$. Hence we obtain the formula $\mathbf{n}\cdot\boldsymbol{\nabla}\mathrm{Tr}K = \mathrm{Tr}K^2$, and furthermore $\mathbf{n}\cdot\boldsymbol{\nabla}\mathrm{Tr}K^2 = 2\mathrm{Tr}K^3$.

[1] M. Kleman, in *Magnetism of Metals and Alloys*, edited by M. Cyrot (North-Holland, Amsterdam, 1982).
[2] K. Kawasaki and T. Ohta, Physica A **118**, 175 (1983).
[3] A. N. Pargellis, P. Finn, J. W. Goodby, P. Panizza, B. Yurke, and P. E. Cladis, Phys. Rev. A **46**, 7765 (1992), and references therein.
[4] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 1995).
[5] A. J. Bray, Adv. Phys. **43**, 357 (1994), and references therein.
[6] H. Tutu and H. Fujisaka, Phys. Rev. B **50**, 9274 (1994).
[7] H. Tutu, Phys. Rev. E **56**, 5036 (1997).

[8] N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979).
[9] I. M. Lifshitz, Zh. Eksp. Teor. Fiz. **42**, 1365 (1962) [Sov. Phys. JETP **15**, 939 (1962)].
[10] S. M. Allen and J. W. Cahn, Acta Metall. **27**, 1085 (1979).
[11] In some papers or books the alternative definition $\nabla\cdot\mathbf{n}=\mathrm{Tr}K$ is used. However, we have interpreted it as $\nabla\cdot\mathbf{n}=\Sigma_\alpha\mathbf{t}_\alpha\cdot(\partial\mathbf{n}/\partial a_\alpha)=-\mathrm{Tr}K$.
[12] A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd ed. (Dover, New York, 1992).